MATHEMATICS

Funky Linear Operators: A Pre-Conference Fever Dream

A. Yodel-Cummings[a], *Fresh and Fruity Sauvignon Blanc 2022*[b]

In which I perform markedly sloppy functional analysis shortly before taking an 'extended sabbatical'.

Introduction

It is just under two weeks until I am obliged to present the findings of my research group at a conference in Cologne. This is to demonstrate to the world what $300,000 in research funding can achieve in a small institution with dubious, yet numerous, financial connections with several medium-sized petrochemical corporations and the Department of Agriculture.

In spite of the pending results of the misconduct investigation regarding the events of the Faculty of Science Christmas Party of 2021, it has been decided that I should be the one to address the conference. This is likely because the other members of the group are either PhD students—who spend 18 hours a day under the influence of research amphetamines, calculating the spectra of bizarre non-linear operators—or my colleague, Darren, who has not come into the office since December for reasons best left unexamined. I have resolved to go, if for no other reason than with the hope of seducing the Head of Department's rather fetching spouse of indeterminate gender.

It is under this perfect storm that I, on my tenth glass of Californian Sauvignon Blanc, begin to write mathematics like all good groundbreaking mathematics is written; straight from my head into LaTeX. Fuck you, I'm not going to check it.

Background

Define by $Q_8 = \langle 1, i, j, k | i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j \rangle$ the quaternion group, and let $\mathbb{Y} = \mathbb{R}[\mathbb{Z}_3 \times Q_8 \times \mathbb{O}]$ be the Yodel-Cummings numbers, a group ring over the reals, where $\mathbb{O}$ is the group of octonions, which I am too off my face to define here. I haven't checked if this is actually a ring, but one time I got a text about something similar from one of the PhD students at 4am, so it's probably fine.

Define the **quaternion conjugate** by $\bar{i} = -i$, $\bar{j} = -j$, $\bar{k} = -k$, $\bar{1} = 1$, $x \in \mathbb{R} \Rightarrow \bar{x} = x$ and $\overline{ab} = \bar{a}\bar{b}$ and $\overline{a+b} = \bar{a} + \bar{b}$ for all $a, b \in \mathbb{R}[Q_8]$. While pretending that $\mathbb{Y}$ is a field—because it would be convenient for the calculations if it was, and I need that severance pay goddammit—we seek to define a generalised inner product $\langle \cdot, \cdot \rangle$ over the vector space $\mathbb{Y}^n$ such that $\mathbb{Y}^n$ is complete with respect to the norm induced by $\langle \cdot, \cdot \rangle$ and so $(\mathbb{Y}^n, \langle \cdot, \cdot \rangle)$ looks kinda like a Hilbert space.

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Figure 1: My only friend following that fateful night.

I think I'm going to call one of those PhDs, I need a tabasco enema.

Underlying Theory

When you combine enough groups together under the Cartesian product—and fudge something about the resultant symmetries appearing at the string level, or something similarly incomprehensible—nobody can visualise the group operations well enough to question you. After all, there was that one time at a regional conference held at CU,NT that some nerd in a hideous bow-tie started rambling about a new non-commutative algebra over $l_2 \times \mathbb{R} \times something$ and everyone believed her without checking. See? I'm going to be like a gauge theorist, but with more crack in my lungs.

Let (X, Q_8) be a space-ring couple (I'll define it once the eye spasms stop) and **F** be a field. We say the bilinear form $\langle \cdot, \cdot \rangle : X \times X \to \mathbf{F}$ is a **generalised inner product** if

- $\langle x, x \rangle \in \mathbb{R}_{\geq 0}$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ for all $x \in X$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in X$
- $\langle \alpha x, z \rangle = \alpha \langle x, z \rangle$ for all $x, z \in X$, $\alpha \in \mathbf{F}$

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$

where $\overline{x}$ is the quaternion conjugate of x.

We seek to find a generalised inner product $\langle \cdot, \cdot \rangle$ such that $\mathbb{Y}^n$ is complete under the norm $\| \cdot \|$ induced by $\langle \cdot, \cdot \rangle$, that is $\|x\| = \sqrt{\langle x, x \rangle}$ for each $x \in \mathbb{Y}^n$. Remember, I am truly talking out of my arse here, so we say that $\mathbb{Y}$ is the field **F** and pretend this is the case until it all breaks and I have to fudge a different solution.

Towards a Result

It turns out it's actually pretty hard to do rigorous mathematical research after a second bottle of wine. I circumvented this issue by posting obviously wrong results on StackExchange and waiting for righteously incensed nerds to correct them and so do my work for me. Unfortunately, there was little of substance for them to correct. Thankfully, I still have plenty of substances left.

You know what, maybe it's fine that $\mathbb{Y}$ is a ring. Sure, why not, it's just another generalisation, right? Pure mathematicians love that, for some reason. What is a ring if not a generalisation of a field? What is a field if not a generalisation of the complex numbers? What is the telephone number of my ex? Maybe this would be easier if I was st–

Let R be a ring and X be a vector space. We call (X, R) a **space-ring couple** if $ax \in X$ (no, I'm not going to define what it means for this to be well-defined, just that it needs to be) for all $a \in R$ and for all $x \in X$. If (X, R) is a space-ring couple, we call the bilinear form $\langle \cdot, \cdot \rangle : X \times X \to R$ a **double generalised inner product** over (X, R) if

- $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ for all $x \in X$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in X$
- $\langle \alpha x, z \rangle = \alpha \langle x, z \rangle$ for all $x, z \in X$, $\alpha \in R$
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$

We also take the time and heart-wrenching effort to generalise linear operators. Let $\{(X, R), (Y, R)\}$ be two space-ring couples with respect to the same ring R. Then we call $T : X \to Y$ a **space-ring linear operator** if $T(ax + by) = aT(x) + bT(y)$ for each $a, b \in R$ and for each $x, y \in X$.

Further Towards a Result

I mean, we need a whacky linear operator now, right? Otherwise I'll have to do my Elmo impression in front of the German corn syrup people.

Note that $\mathbb{Y}$ is a vector space over $\mathbb{R}$. Or, even better, that $(\mathbb{Y}, \mathbb{Y})$ is a space-ring couple. What about $(\mathbb{Y}[x]^2, \mathbb{Y})$? Cor, I can decorate blackboard bold letters with brackets and indices and it can mean whatever I like. In this case, though, $\mathbb{Y}[x]$ denotes—as you would expect—the ring of polynomials over $\mathbb{Y}$ with coefficients in $\mathbb{Y}$. Isn't research wonderful?

Now, define $\Gamma : \mathbb{Y}[x]^2 \to \mathbb{Y}[x]^2$ by $\Gamma((f(x), g(x))) = \left(\frac{d^2 f(x)}{d(x^2)^2}, \frac{d^{0.2}(f(x)+g(x))}{dx^{0.2}} \right)$ where $f, g \in \mathbb{Y}[x]$. Unfortunately, $\mathbb{Y}[x]^2$ is not complete under any reasonable metric I can think of, let alone one from a norm induced by an inner product. Perhaps I'll think of one tomorrow after sufficient complimentary research chemicals.

Key Results

I think I might be able to sleep now.

Conclusion

I need some water.

Acknowledgements

I did this groundbreaking research all by myself, but I would especially like to thank Schilbux Petromotive for generously covering my chemical expenses over the course of my short yet storied career.